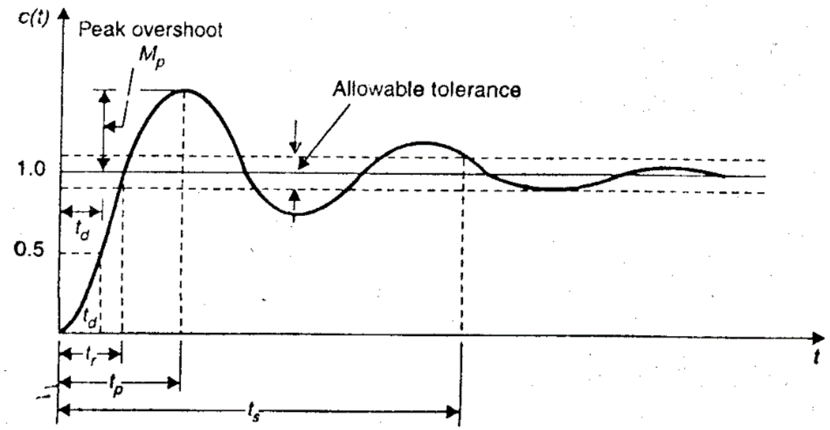
ELEC 2400 Electronic Circuits

Chapter 6: Transient Analysis



Course Website: https://canvas.ust.hk

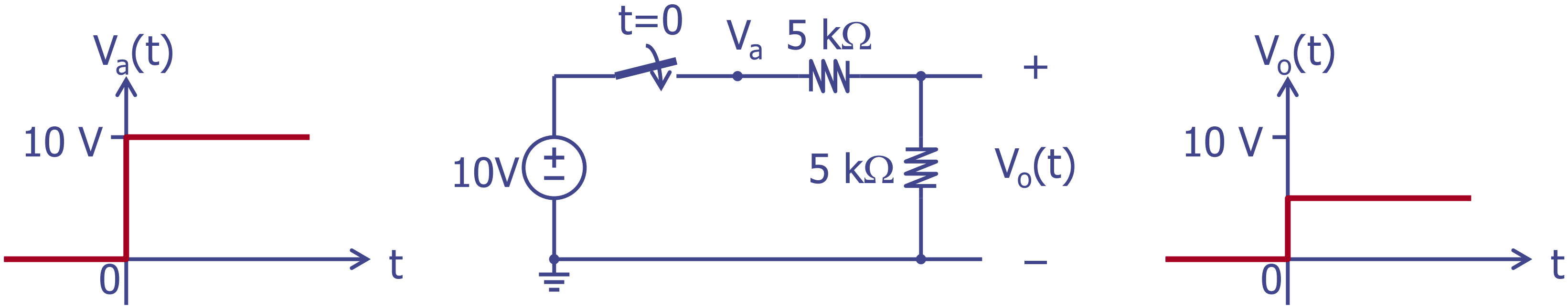
HKUST, 2021-22 Fall

Chapter 6: Transient Analysis

6.1 Circuit Dynamics

A resistive network has trivial circuit dynamics: all voltages and currents are linear functions of (that is, proportional to) input sources.

Example 6-1: The output voltage of a resistor divider circuit follows the input voltage exactly.



However, many applications need energy storing elements, such as capacitors and inductors, along with switching actions that has specific circuit dynamics to achieve useful functions. The study of circuit dynamics is also known as transient analysis.

Applications of Circuit Dynamics

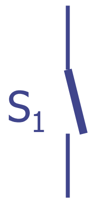
A security alarm circuitry makes use of circuit dynamics: when you enter your apartment, you have to key in the pass code within, say, 30 seconds; otherwise, the alarm may set off.

Another example: if an intruder of your apartment has triggered the sensor of the alarm system, the alarm may set off after 30 seconds.

Very often, the timer consists of an RC (resistor + capacitor) circuit: an RC timer.

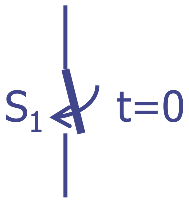
Switches and Operations

A switch has two states: open and closed. For an ideal switch, when it is closed, the resistance is zero; and when it is open, the resistance is infinite.

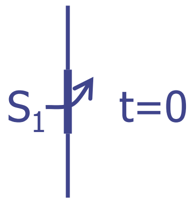


switch open

switch closed



switch closed at t=0: for $t < 0$, S_1 open; for $t \geq 0$, S_1 closed.



switch open at t=0: for $t < 0$, S_1 closed; for $t \geq 0$, S_1 open.

6.2.1 Charging Capacitor with Current Source

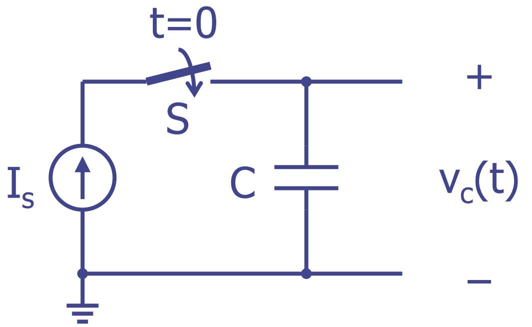
Recall the time-domain relation of a capacitor:

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

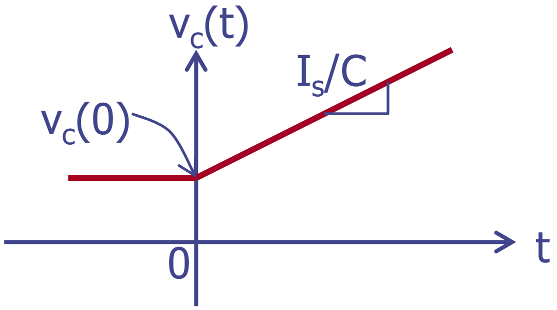
Now, consider a capacitor C being charged by a constant current source I_s through a switch S:

$$\frac{dv_c(t)}{dt} = \frac{I_s}{C}$$

$$v_c(t) = \frac{I_s}{C}t + v_c(0)$$

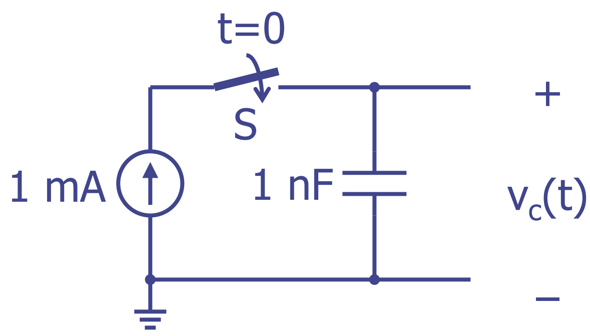


The time response of $v_c(t)$ is as shown:



Example 6-2

Example 6-2: A current source of 1 mA is used to charge up a capacitor of 1 nF with $v_c(0)=0$. How long does it take to charge the capacitor to 5 V if it is initially relaxed?



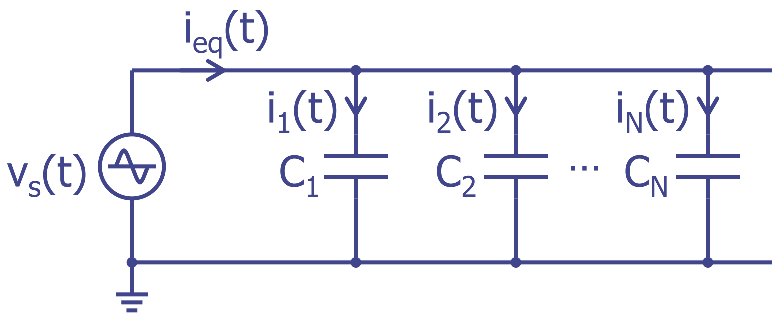
Soln.:

$$I_s = C\frac{\Delta V_c}{\Delta t}$$

$$\Rightarrow \quad \Delta t = \frac{C\Delta V_c}{I_s} = \frac{1nF \times 5V}{1mA} = 5\mu s$$

6.2.2 Capacitors in Parallel and in Series

Consider driving N capacitors connected in parallel with a time varying voltage source $v_s(t)$:



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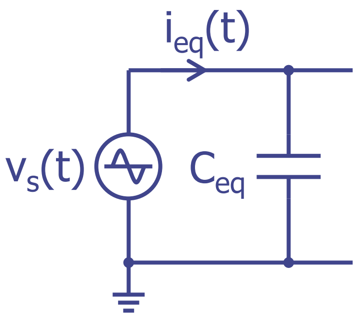


$$i_{eq}(t) = i_1(t) + i_2(t) + \ldots + i_N(t)$$

$$i_{eq}(t) = (C_1 + C_2 + \ldots + C_N)\frac{dv_s(t)}{dt}$$

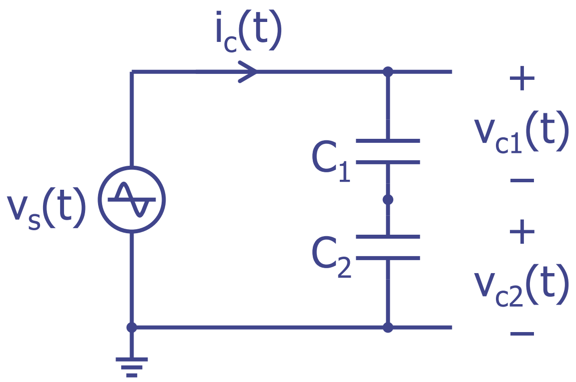
$$i_{eq}(t) = C_{eq}\frac{dv_s(t)}{dt}$$

Hence, for capacitors connected in parallel:

$$C_{eq} = C_1 + C_2 + \ldots C_N$$

Capacitors in Series

Consider driving two capacitors connected in series with a time varying voltage source $v_s(t)$:Type equation here.



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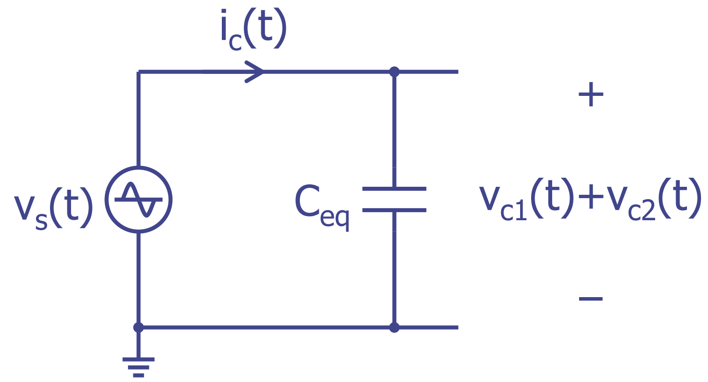


$$v_s(t) = v_{c1}(t) + v_{c2}(t)$$

$$\frac{dv_s(t)}{dt} = \frac{dv_{c1}(t)}{dt} + \frac{dv_{c2}(t)}{dt}$$

$$i(t) = C\frac{dv(t)}{dt}$$

$$\therefore \quad \frac{i_c(t)}{C_{eq}} = \frac{i_c(t)}{C_1} + \frac{i_c(t)}{C_2}$$

$$\Rightarrow \quad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Capacitors in Series (cont.)

For two capacitors connected in series, we therefore have

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{C_1 + C_2}{C_1 C_2}$$

$$C_{eq} = C_1 \,||\, C_2$$

Note that a||b is only a symbol that represents the computation of

$$a \,||\, b = \frac{ab}{a+b}$$

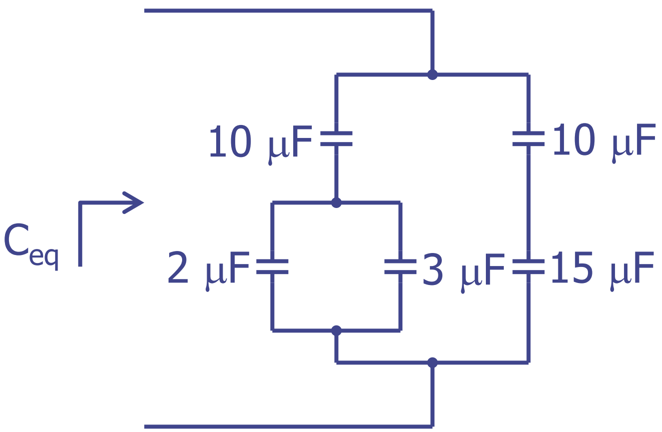
In general, for N capacitors connected in series, we have

$$C_{eq} = C_1 \,||\, C_2 \,||\, \ldots \,||\, C_N$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}$$

Example 6-3

Example 6-3: Compute the equivalent capacitance C_{eq} of the following capacitor network.



Soln.:

$$C_{eq} = 10\mu || (2\mu + 3\mu) + 10\mu || 15\mu$$

$$= \frac{10\mu \times 5\mu}{10\mu + 5\mu} + \frac{10\mu \times 15\mu}{10\mu + 15\mu}$$

$$= 3.33\mu + 6\mu$$

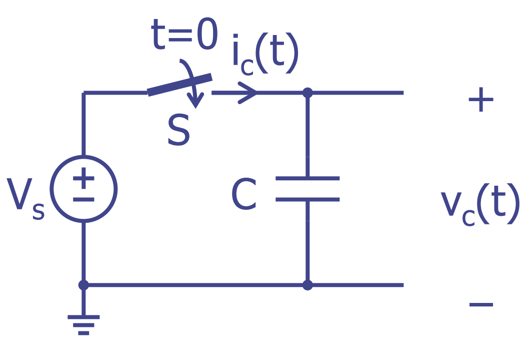
$$= 9.33\mu F$$

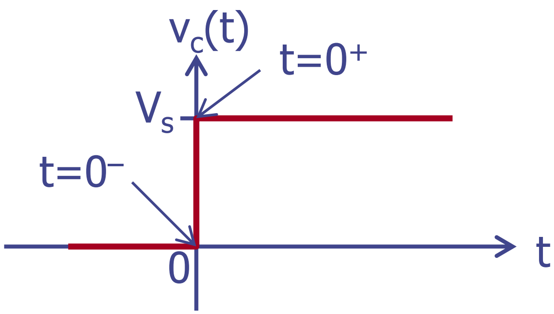
6.2.3 Charging Capacitor with Voltage Source

Charging a capacitor C (initially relaxed, or $v_c(0)=0$) with a constant voltage source V_s is not as straightforward. As the voltage source forces the voltage across itself to be V_s, the capacitor is charged to V_s instantaneously:

$$V_o(0^-) = 0$$
$$V_o(0^+) = V_s$$

where $t=0^-$ is the time instant just right before switching, and $t=0^+$ is the time instant just right after switching, with the time elapsed being $\Delta t = 0^+ - 0^- = 0$.





6.2.4 Continuity of Capacitor Voltage

The capacitor current I_c at the previous charging process is thus

$$I_c = \frac{\Delta Q}{\Delta t} = \frac{C\Delta V_c}{\Delta t} = \frac{C(v_c(0^+) - v_c(0^-))}{0} = \frac{CV_s}{0} = \infty$$

Mathematically I_c is a delta function ($i_c(t) = \delta(t)$, not discussed in this course); but practically it cannot be achieved. Hence, if the capacitor current $i_c(t)$ remains finite, the capacitor voltage $v_c(t)$ cannot have jumps in voltage. We called this condition the continuity of the capacitor voltage: no jumps in capacitor voltage, i.e. $v_c(0^+) = v_c(0^-)$.

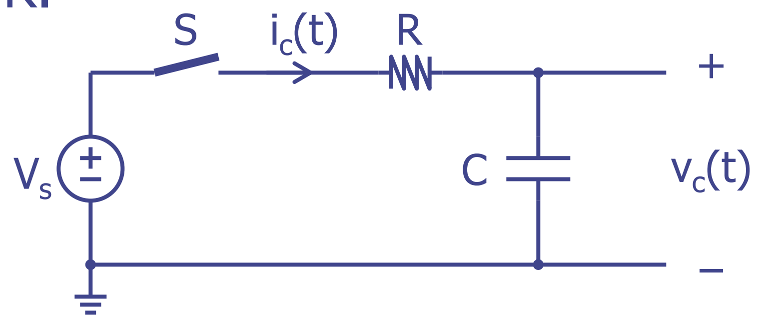
Another way to prove the continuity of the capacitor voltage is by the energy method. The energy stored in a capacitor is

$$E = \tfrac{1}{2}CV^2$$

A sudden jump in voltage will lead to a sudden jump in energy, which implies infinite power. This is physically impossible.

Charging Capacitor with V_s through R

In reality, a voltage source will have a non-zero source resistance, and the switch will have a non-zero switch resistance when closed. The two resistors are in series, and can be modeled as a single resistor R.



Consider the general case that the capacitor C has an initial voltage of $v_c(0)$ (not necessarily zero). At t=0, the switch S is closed, and the capacitor current $i_c(t)$ is given by

$$\frac{V_s - v_c(t)}{R} = i_c(t) = C\frac{dv_c(t)}{dt}$$

6.2.5 First Order Differential Equation

Rearranging the terms and assign the time constant τ=RC, we arrive at the first order ordinary differential equation (ODE):

$$\frac{dv_c(t)}{dt} + \frac{v_c(t)}{\tau} = \frac{V_s}{\tau}$$

Solving the above equation requires multiplying both sides with the integration factor $e^{t/\tau}$:

$$\frac{dv_c(t)}{dt} e^{t/\tau} + \frac{v_c(t)}{\tau} e^{t/\tau} = \frac{V_s}{\tau} e^{t/\tau}$$

$$\Rightarrow \qquad \frac{d\left(v_c(t)e^{t/\tau}\right)}{dt} = \frac{V_s}{\tau} e^{t/\tau}$$

$$\int_{v_c(0^+)}^{v_c(t)e^{t/\tau}} d\left(v_c(t')e^{t'/\tau}\right) = \int_{0^+}^{t} \frac{V_S}{\tau} e^{t''/\tau} dt''$$

Solution to First Order ODE

The (general) solution to the first order ODE is

$$v_c(t)e^{t/\tau} - v_c(0^+) = V_S e^{t/\tau} - V_S$$

which can be rearranged to read

$$v_c(t) = v_c(0^+)e^{-t/\tau} + V_S - V_S e^{-t/\tau} \;= V_s + (v_c(0^+) - V_s)e^{-t/\tau}$$

We next observe that

$$v_c(\infty) = V_s$$

and the solution is more conveniently written as

$$v_c(t) = v_c(\infty) + ((v_c(0^+) - v_c(\infty))e^{-t/\tau}$$

In words, the solution is

transient = final + (initial – final) $e^{-t/\tau}$

Two Equivalent Formulas For Transients

First formula from previous page

$$\text{transient} = \text{final} + (\text{initial} - \text{final})\, e^{-t/\tau}$$

Rearranging terms

$$\begin{aligned} \text{transient} &= \text{final} - (\text{final} - \text{initial})\, e^{-t/\tau} \\ &= \text{final}\,(1 - e^{-t/\tau}) + \text{initial} \times e^{-t/\tau} \\ &= \text{initial} + \text{final}\,(1 - e^{-t/\tau}) - \text{initial} + \text{initial} \times e^{-t/\tau} \end{aligned}$$

Second formula

$$\text{transient} = \text{initial} + (\text{final} - \text{initial})(1 - e^{-t/\tau})$$

These two formulas are equivalent. Feel free to use the one that is more convenient for the particular transient problem.

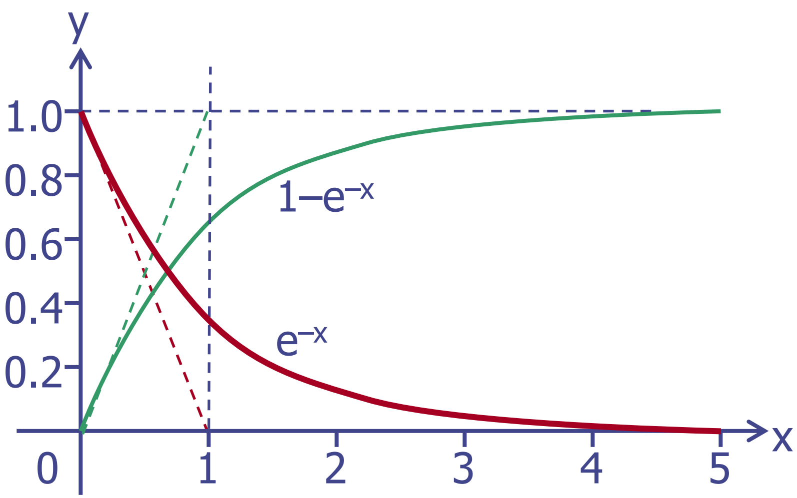
Exponential Decay and Complementary Exponential

Consider the functions involving exponentials with a negative index, that is, with $x>0$, we consider

$y = e^{-x}$ exponential decay

and $y = 1 - e^{-x}$ complementary exponential

x	$y = e^{-x}$	$y = 1-e^{-x}$
0	1.000	0.000
0.693	0.500	0.500
1	0.368	0.632
2	0.135	0.865
2.3	0.100	0.900
3	0.050	0.950
4	0.018	0.982
4.6	0.010	0.990
5	0.007	0.993
6	0.002	0.998
6.9	0.001	0.999



Time Constant vs Half Life

In sketching the exponential decay curve (or complementary exponential curve), three parameters should be indicated:

Initial value
Final value
Time constant

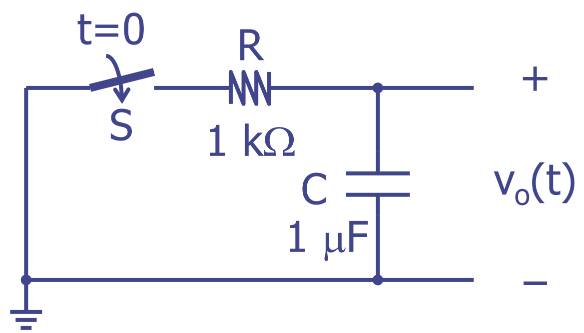
The time constant τ is a circuit parameter. Readers may learn that radioactive decay also obeys the exponential decay curve, but it is specified by the parameter half life $T_{1/2}$, the time for the original mass of radioactive component to reduce by one-half.

From p. 6-17, the original value 1 is reduced to ½ at x=0.693, and this is translated to $0.693\tau \approx 0.7\tau$ in p. 6-16. Hence, roughly speaking,

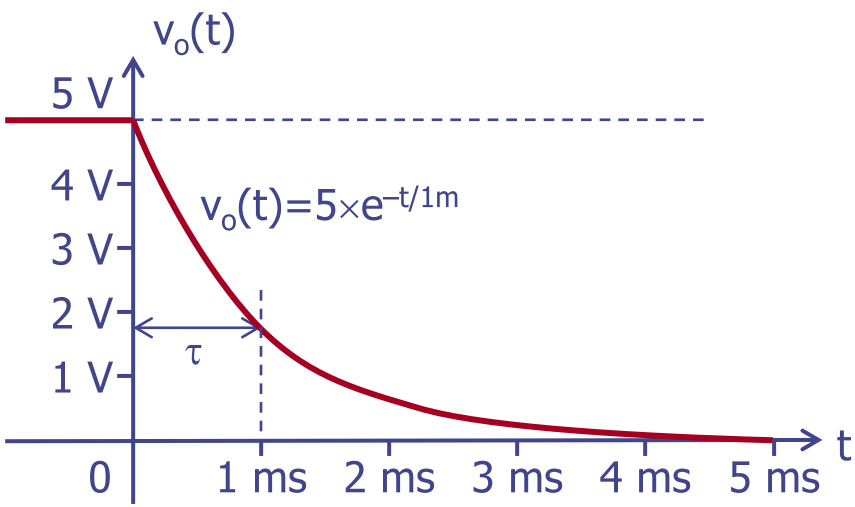
$$T_{1/2} = 0.7\tau.$$

Example 6-4: RC Circuit Transient

Example 6-4: The capacitor C was initially charged to 5 V. Sketch the capacitor voltage $v_o(t)$ when it is discharged through the resistor R for $t \geq 0$.



Soln.:
Initial value = $v_o(0^+)$ = 5 V
Final value = $v_o(\infty)$ = 0 V
Time constant = τ = RC
= 1 kΩ×1 μF = 1 ms



The sketch is a curve of exponential decay.

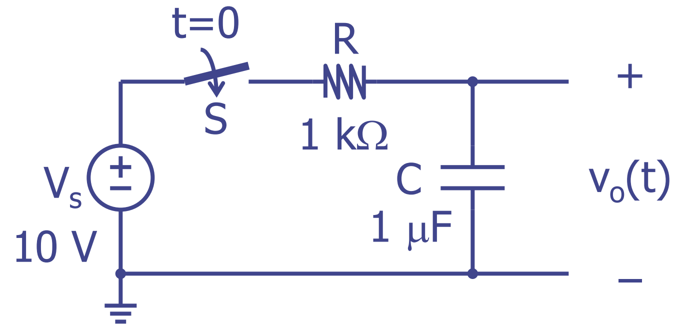
Example 6-5: RC Circuit Transient

Example 6-5:

(a) Sketch the output voltage $v_o(t)$ of the RC circuit shown below, assuming the capacitor is initially relaxed.

(b) How long does it take for the output voltage $v_o(t)$ to reach 99% of the final value?

(c) Sketch $v_o(t)$ if $v_o(0^-) = 4$ V.



Soln.:

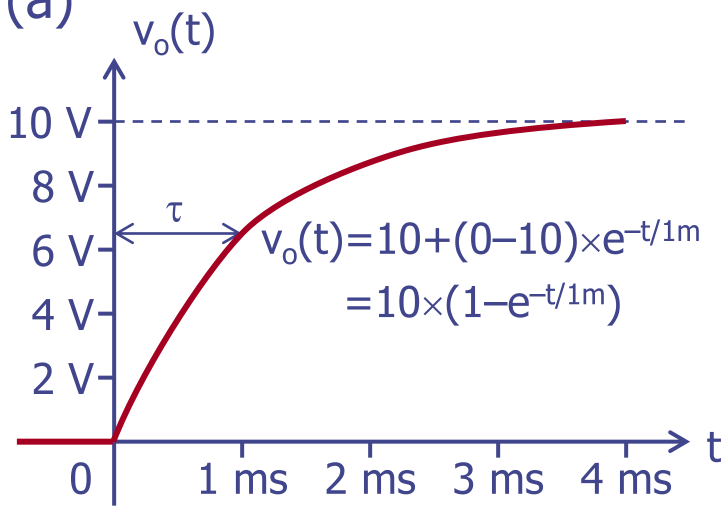
(a) Initial value $= v_o(0^+) = 0$ V

Final value $= v_o(\infty) = V_s = 10$ V

Time constant $= \tau = RC = 1\ \text{k}\Omega \times 1\ \mu\text{F} = 1$ ms

Example 6-5: RC Circuit Transient (cont.)

(a)



(b) For $v_o(t)$ to reach 0.99 of the final value, we have

$$v_o(t) = V_s(1 - e^{-t/\tau})$$

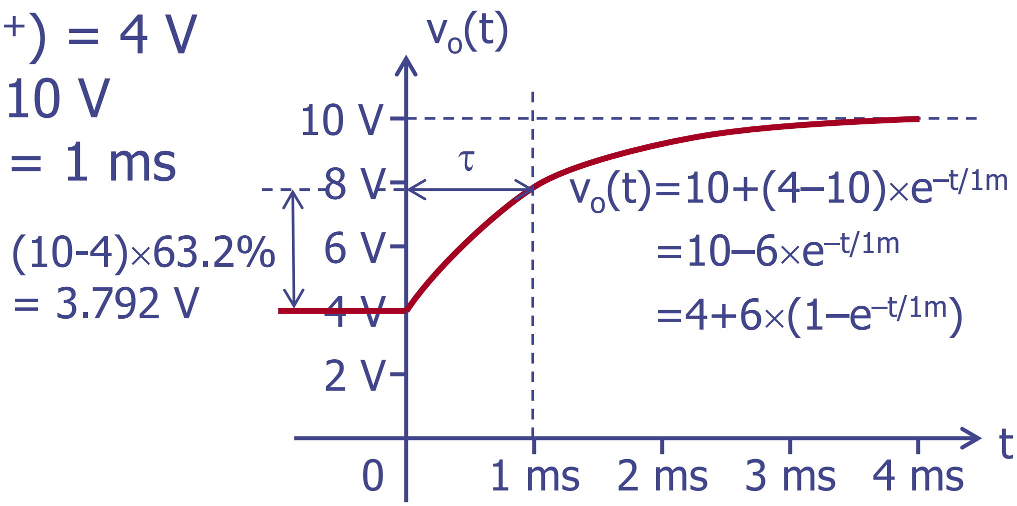
$$\Rightarrow \quad 0.99V_s = V_s(1 - e^{-t/\tau})$$

$$\Rightarrow \quad t = -\tau\times\ln(0.01)$$

$$= 1 \text{ ms}\times 4.6$$

$$= 4.6 \text{ ms}$$

(c) Initial value = $v_o(0^+)$ = 4 V
Final value = V_s = 10 V
Time constant = τ = 1 ms



6.2.6 Capacitor as Open Circuit in Steady State

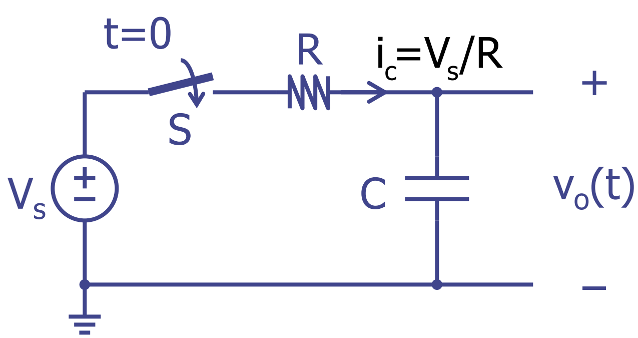
For the RC circuit, let us consider the capacitor current $i_c(t)$:

$$i_c(t) = C\frac{dv_c(t)}{dt} = C\frac{d([v_c(\infty)+(v_c(0^+)-v_c(\infty))e^{-t/\tau}])}{dt}$$

$$= \frac{v_c(\infty)-v_c(0^+)}{R}e^{-t/\tau} = \frac{V_S}{R}e^{-t/\tau}$$

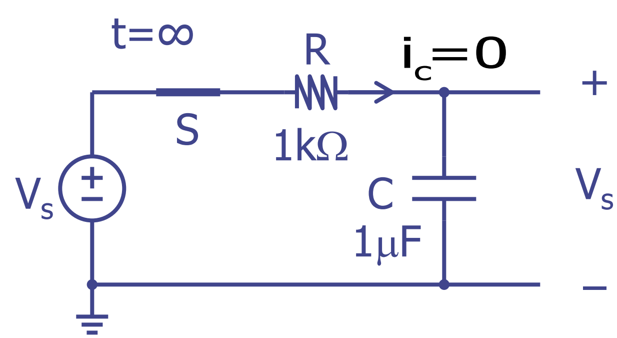
In the initial state ($t=0^+$), with continuity of the capacitor voltage $v_c(0^+)=v_c(0^-)=0$:

$$i_c(0^+) = \frac{V_s}{R}$$



In the final state ($t=\infty$), the capacitor behaves as an open circuit (for $i_c(\infty)=0$):

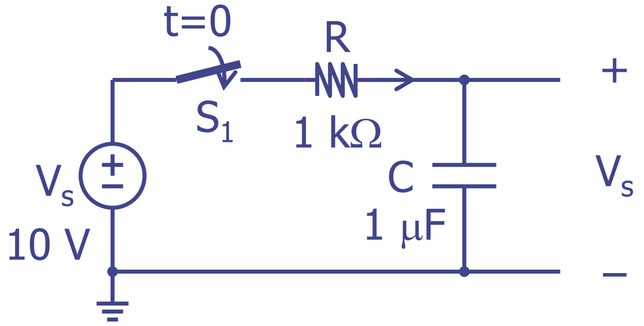
$$i_c(\infty) = 0$$



Energy Balance in RC Circuit

Example 6-6: By fully charging the capacitor C to V_s:

(a) What is the energy stored in C?
(b) What is the energy delivered by V_s?
(c) What is the energy consumed (dissipated) by R?



Soln.:

(a) The energy stored in C is

$$E_C = \frac{1}{2} C V_s^2 = \frac{1}{2} \times 1\mu \times 10^2 = 50\mu J$$

(b) The energy delivered by V_s is

$$E_S = \int_0^\infty \left(V_s \times -i_s(t)\right) dt \quad \text{(–ve sign to indicate delivering out)}$$

$$= \int_0^\infty \left(V_s \times \frac{V_s}{R} e^{-t/CR}\right) dt$$

Energy Balance in RC Circuit (cont.)

$$E_S = CV_s^{\,2}\left[-e^{-t/CR}\right]_0^{\infty} = CV_s^{\,2}$$

$$= 1\mu \times 10^2 = 100\mu J$$

(c) The energy dissipated by R is

$$E_R = \int_0^{\infty}\left(i_c(t)^2 R\right)dt$$

$$= \int_0^{\infty}\left(\frac{V_s^{\,2}}{R^2}e^{-2t/CR} \times R\right)dt$$

$$= \frac{1}{2}CV_s^{\,2}\left[-e^{-2t/CR}\right]_0^{\infty}$$

$$= 50\mu J$$

It is important to note that:

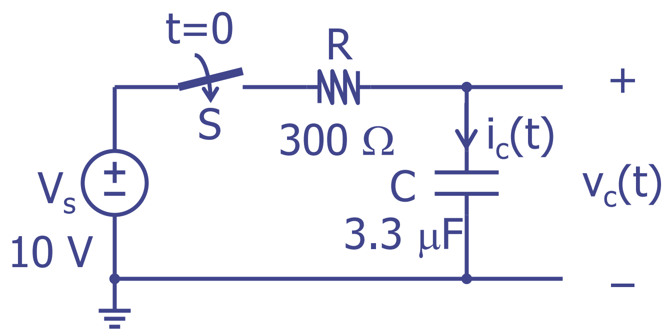
$E_S = E_C + E_R$

Hence, energy is balanced in the charging process of the RC circuit.

Capacitor Current can be Discontinuous

Although capacitor voltage is continuous, capacitor current can be discontinuous, as shown in the following example.

Example 6-7: Sketch the output voltage and the output current waveform if $v_c(0) = 4$ V.



Soln.:

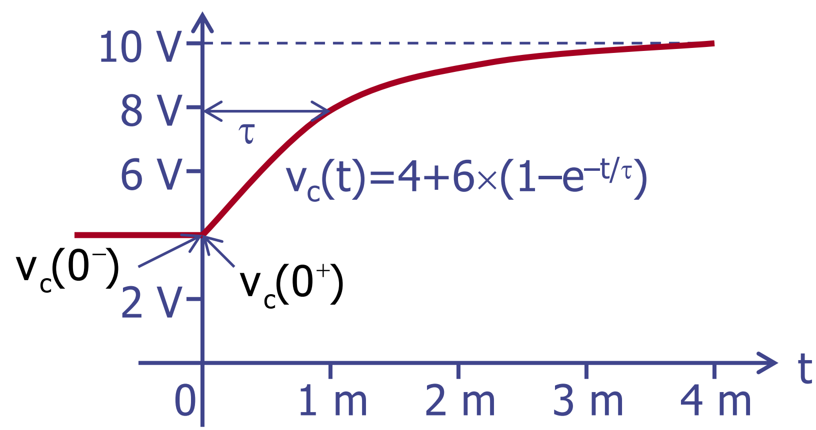
$v_c(0^+) = 4$ V (given)

$v_c(\infty) = 10$ V

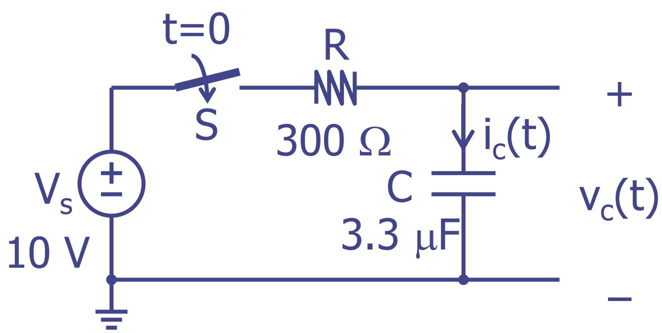
$\tau = 300 \times 3.3\mu = 0.99$ ms ≈ 1 ms

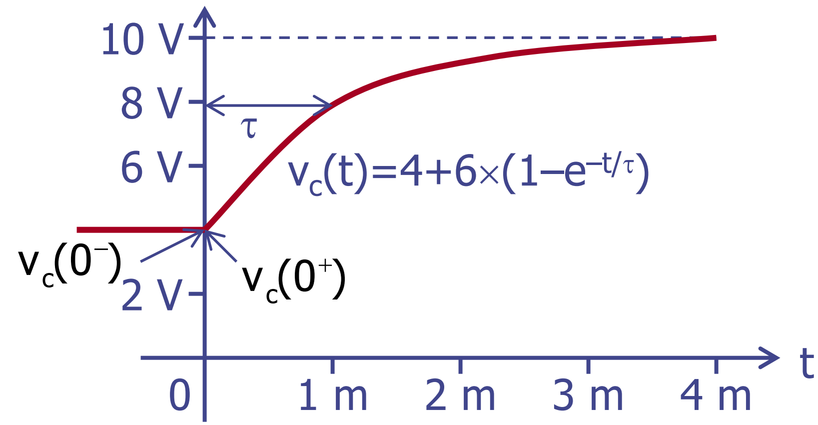
$v_c(t) = 10 + (4 - 10) \times e^{-t/\tau}$

$= 4 + 6 \times (1 - e^{-t/\tau})$



Capacitor Current can be Discontinuous (cont.)





Soln.:

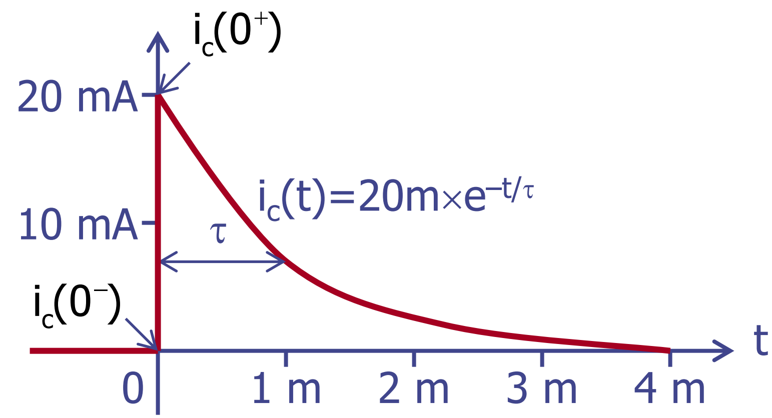
$i_c(0^+) = (10-4)/300 = 20\text{ mA}$

$i_c(\infty) = 0\text{ A}$

$$i_c(t) = \frac{V_s - v_c(t)}{R}$$

$$= \frac{10 - [4 + 6 \times (1 - e^{-t/\tau})]}{300}$$

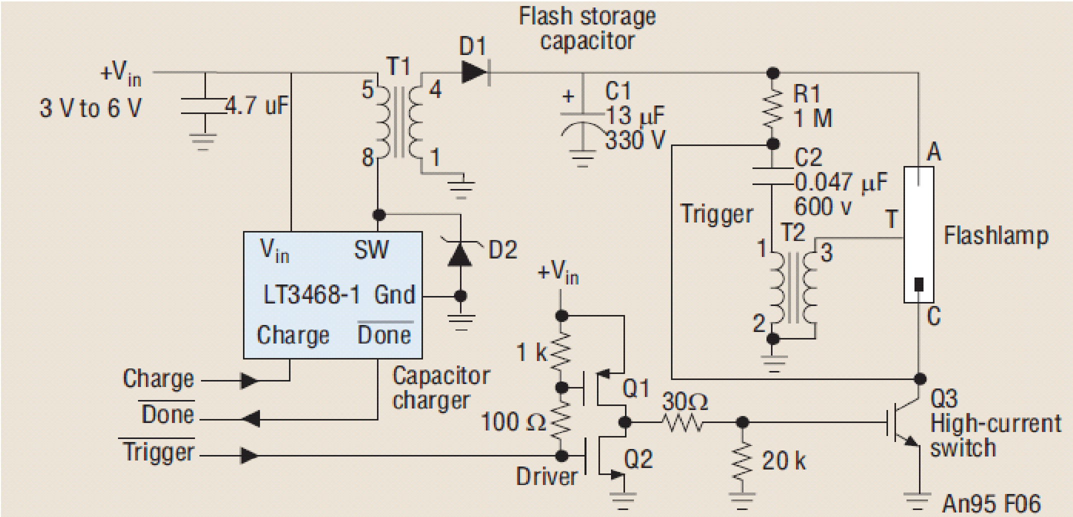
$$= \frac{6 \times e^{-t/\tau}}{300} = 20\text{m} \times e^{-t/\tau}$$



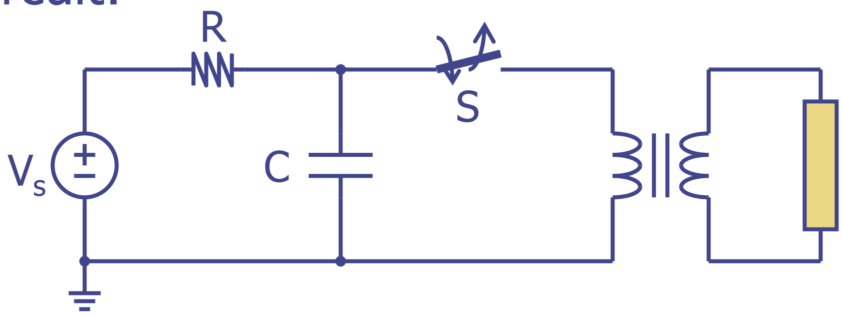
Alternatively, $i_c(t) = i_c(\infty) + (i_c(0^+) - i_c(\infty))e^{-t/\tau} = 20\text{m}\times e^{-t/\tau}$

Application: Flash Lamp Discharge

A camera flash lamp makes use of pulsed capacitive discharge.

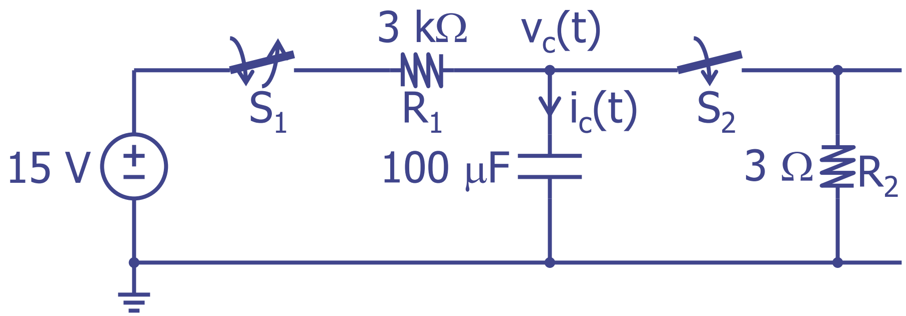


Simplified circuit:



Example 6-8

Example 6-8: Simplified scheme of charge and discharge of flash lamp: close S_1 to charge; (open S_1 and close S_2) to discharge.



Charging scenario (consider t=0 when S_1 is closed):

Initial value: $v_c(0^+) = 0$ V $\quad i_c(0^+) = (15-0)/3k) = 5$ mA

Final value: $v_c(\infty) = 15$ V $\quad i_c(\infty) = 0$ mA

Time constant: $R_1C = \tau_1 = 3k \times 100\mu = 300$ ms

$$v_c(t) = 15(1 - e^{-t/300m})$$

$$i_c(t) = 5m \times e^{-t/300m}$$

Example 6-8 (cont.)

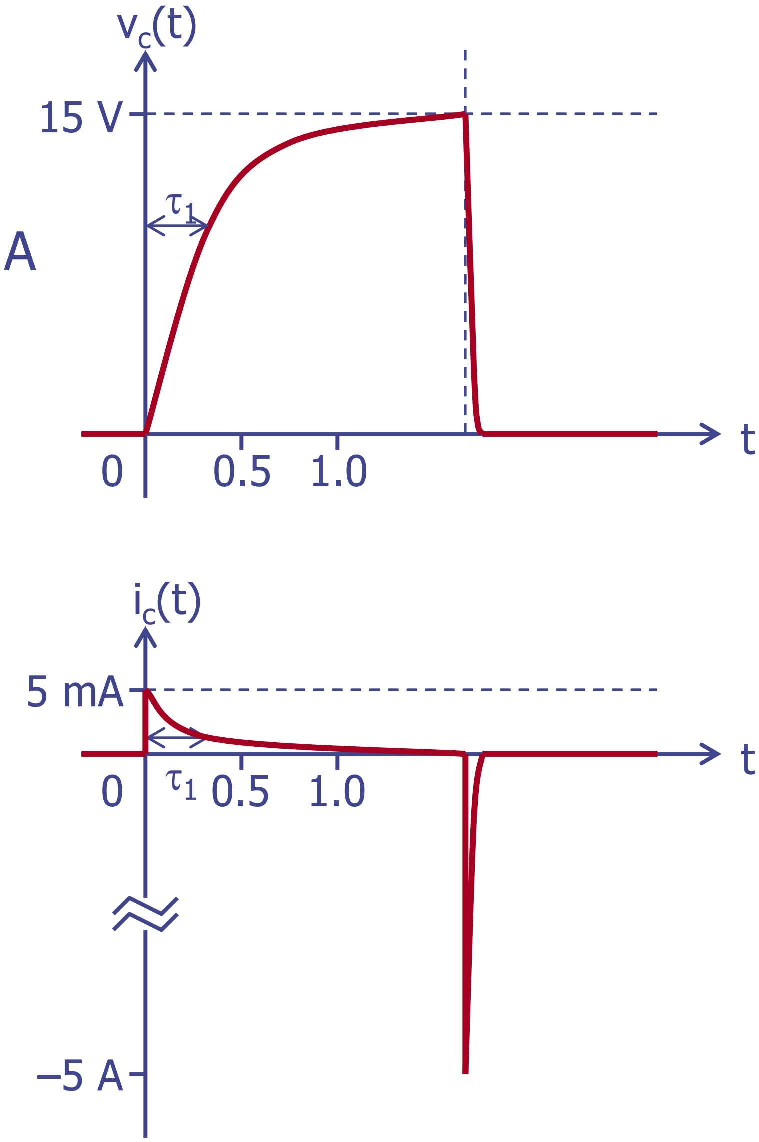
Discharging scenario: (consider t=0 when S_2 is closed):

$v_c(0^+) = 15$ V $\quad i_c(0^+) = -15/3 = -5$ A
$v_c(\infty) = 0$ V $\quad i_c(\infty) = 0$ mA
$R_2C = \tau_2 = 3\times100\mu = 300\ \mu s$

$$v_c(t) = 15e^{-t/300\mu}$$

$$i_c(t) = -5\times e^{-t/300\mu}$$

The charging current is 5 mA with a slow time constant of 0.3 s; while the discharging current is much larger at 5 A and the time constant is much shorter at 300 µs.

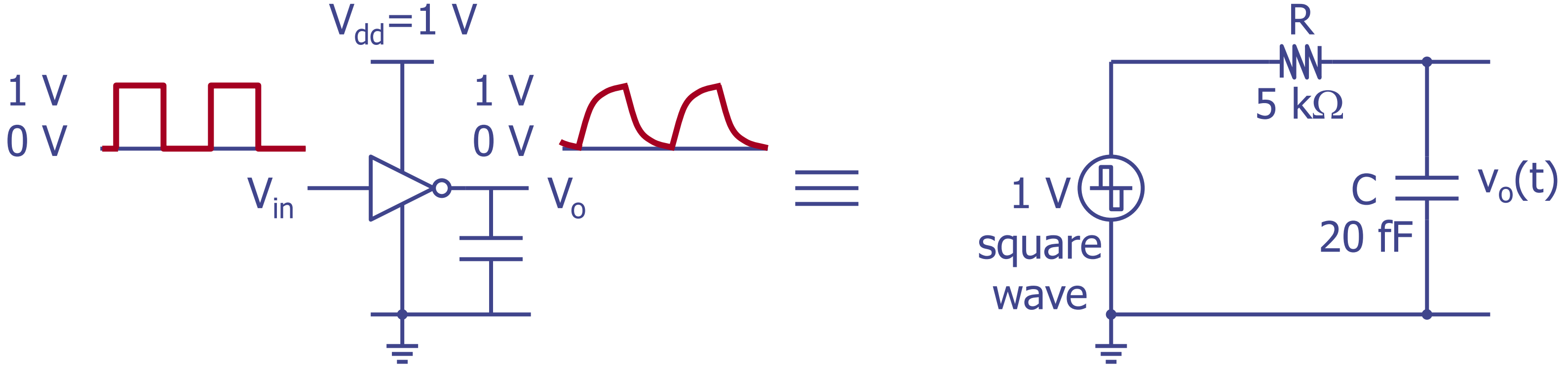


Application: Speed of Computer

The heart of a computer is the CPU (central processing unit) that consists of millions of logic gates. The speed of a computer is determined by the propagation delays of logic gates (modeled as RC circuits) that restrict the maximum switching frequency.

Example 6-9: Given a logic inverter of a CPU that is modeled as an RC circuit with R=5 kΩ and C=20 fF driven by a 1-V square wave, determine the maximum switching frequency f_s if we require the output swing to be better than 0.01 V and 0.99 V.

Soln.:



Example 6-9 (cont.)

For V_{dd}=1 V, the output voltage of the inverter swings between 0 V and 1 V. From previous discussion we learn that if we allow a duration of 5τ (5RC), then the capacitor (output) voltage can settle to within 1% of the final values.

Now,

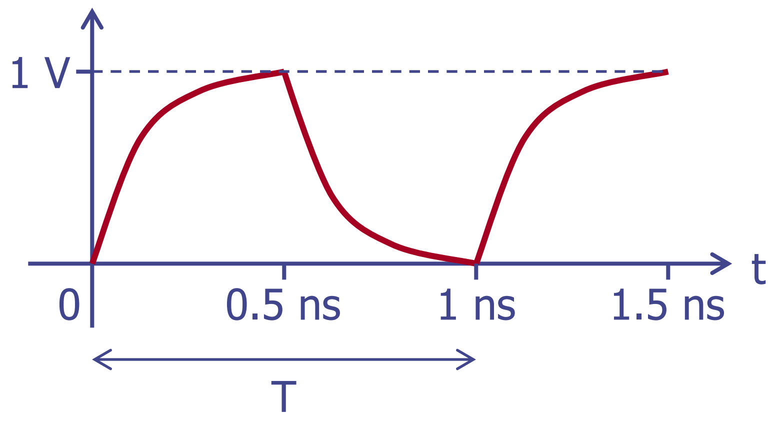
$$\tau = RC = 5k \times 20f$$
$$= 0.1ns$$
$$5\tau = 0.5ns$$

This 0.5 ns is half of the switching period. Therefore,

$$T = 2 \times 5\tau = 1ns$$

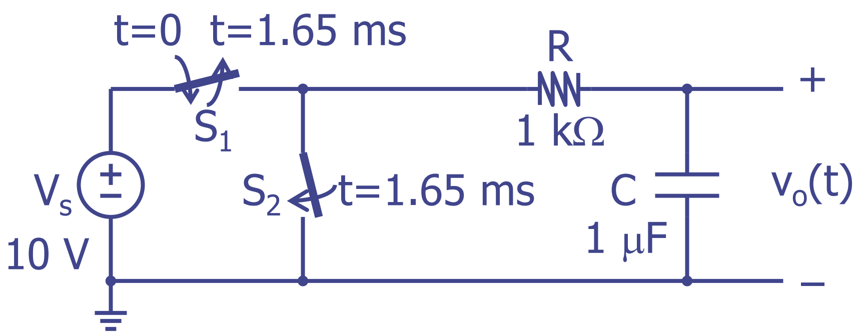
and

$$f_s = 1/T = 1GHz$$

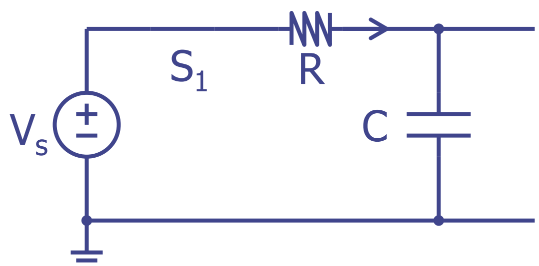


Example 6-10: RC Circuit Transient

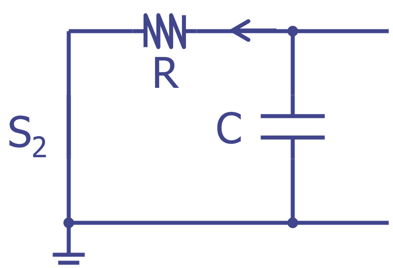
Example 6-10: Sketch the output voltage $v_o(t)$ from t=0 to t=4 ms.

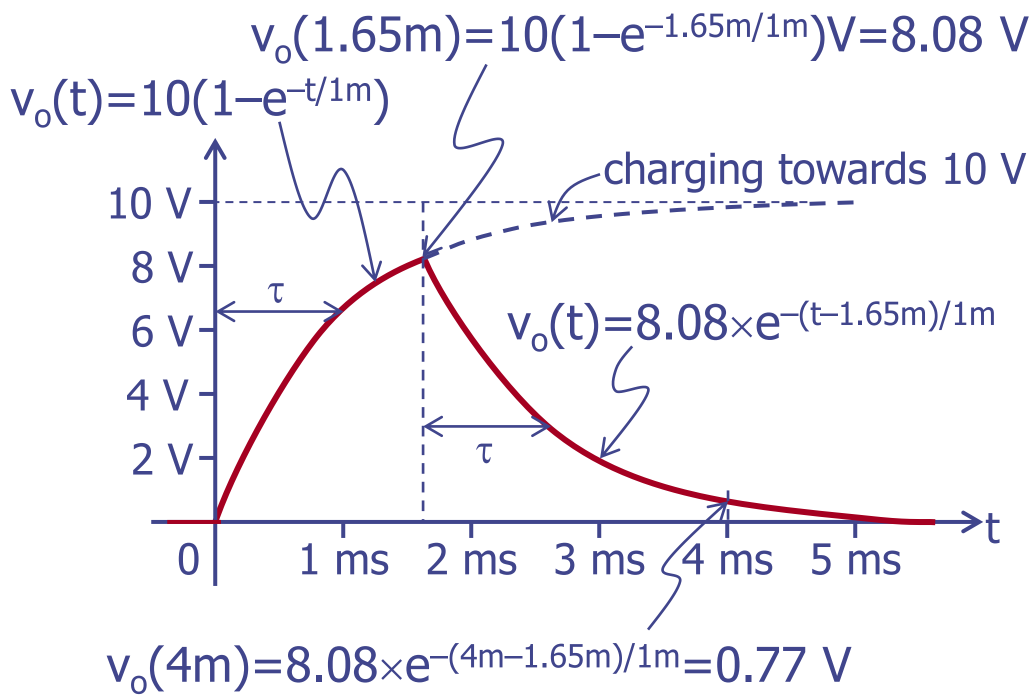


Soln.: For 0 < t < 1.65 ms



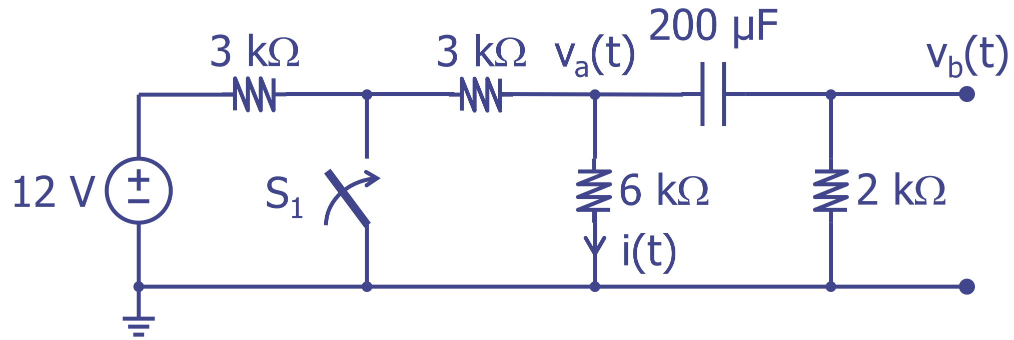
For t > 1.65 ms





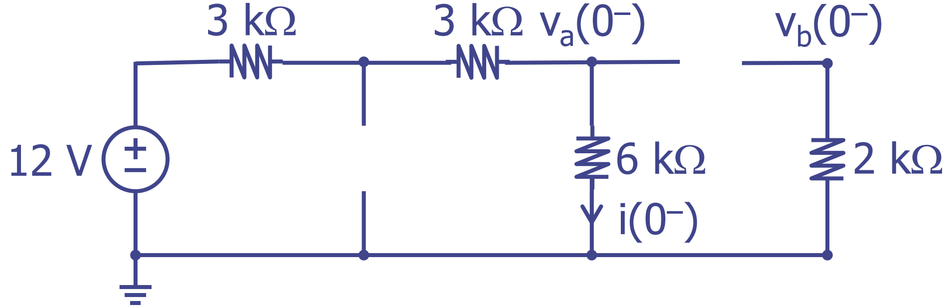
Example 6-11

Example 6-11: The capacitor is not connected to ground in this example. Find expressions for and sketch $v_a(t)$, $v_b(t)$ and $i(t)$ before and after the switch is closed.



Consider t=0 when S_1 is closed. The problem is advanced enough that we need to break it down into 3 stages for analysis.

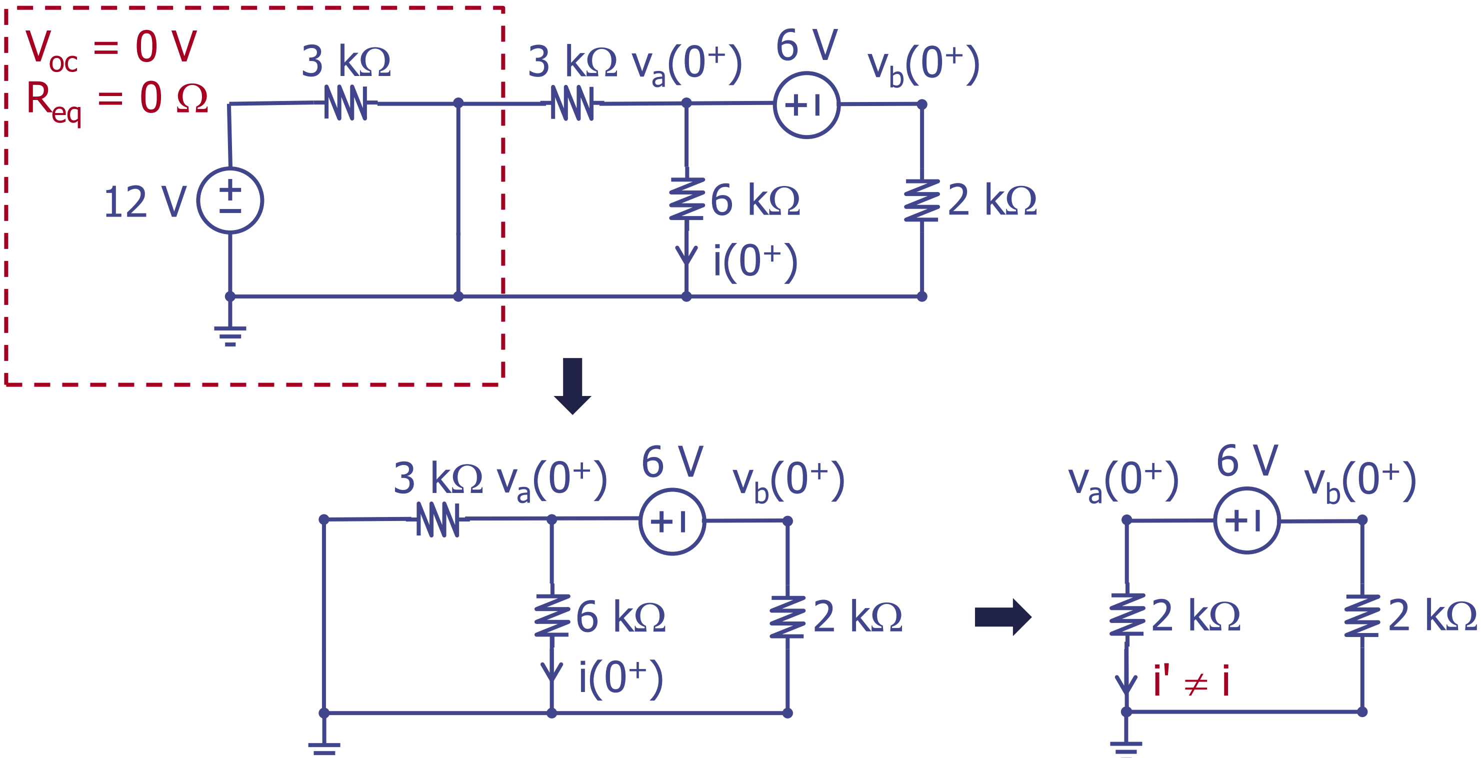
(i) Before the switch is closed, the capacitor behaves as an open circuit.



$v_a(0^-) = 6$ V
$v_b(0^-) = 0$ V
$i(0^-) = 1$ mA

Example 6-11 (cont.)

(ii) $t=0^+$ scenario: the capacitor voltage can't change instantly. Therefore $v_a(0^+) - v_b(0^+) = 6$ V and the capacitor momentarily behaves as a 6-V battery as shown in below.



Therefore

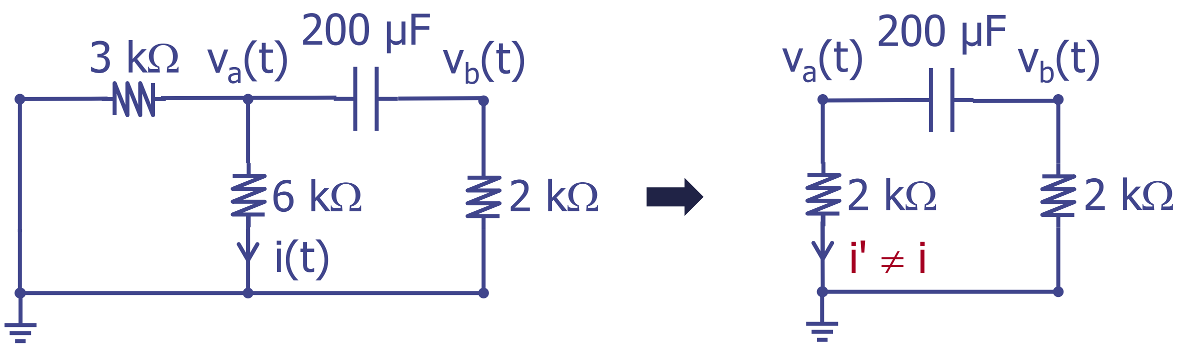
$$v_a(0^+) = 3 \text{ V}$$

$$v_b(0^+) = -3 \text{ V}$$

$$i(0^+) = v_a(0^+)\,/6 \text{ k}\Omega = 0.5 \text{ mA}$$

Example 6-11 (cont.)

(iii) From $t=0^+$ on: the capacitor is fully discharged through a 4 kΩ equivalent resistance.



Initial value: $v_a(0^+) = 3$ V $v_b(0^+) = -3$ V $i(0^+) = 0.5$ mA

Final value: $v_a(\infty) = 0$ V $v_b(\infty) = 0$ V $i(\infty) = 0$ mA

Time constant: $\tau = 4k \times 200\mu = 800$ ms

Therefore

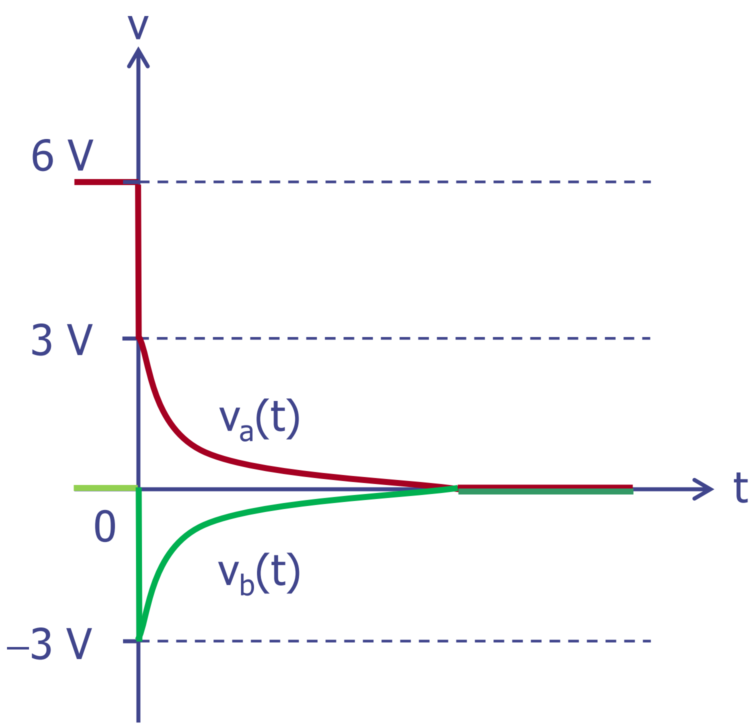
$$v_a(t) = 3e^{-t/800m}$$

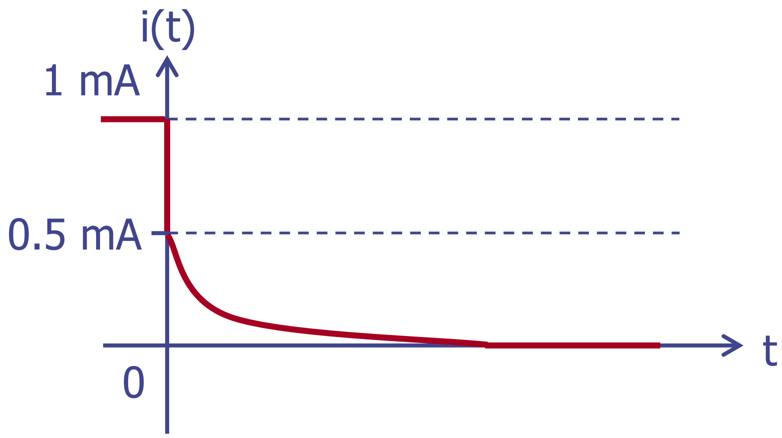
$$v_b(t) = -3e^{-t/800m}$$

$$i(t) = 0.5me^{-t/800m}$$

Example 6-11 (cont.)

We now have all we need to sketch $v_a(t)$, $v_b(t)$ and $i(t)$.





When the switch is closed, both $v_a(t)$ and $v_b(t)$ jerk down by 3 V while the voltage across the capacitor stays momentarily constant. Notice that part of the circuit can acquire a transient voltage that exceeds the supply voltages, i.e., above V_s or below ground.

6.3.1 Charging Inductor with Voltage Source

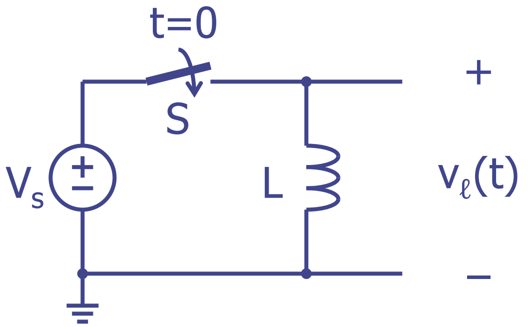
Recall the time-domain relation of an inductor:

$$v_\ell(t) = L\frac{di_\ell(t)}{dt}$$

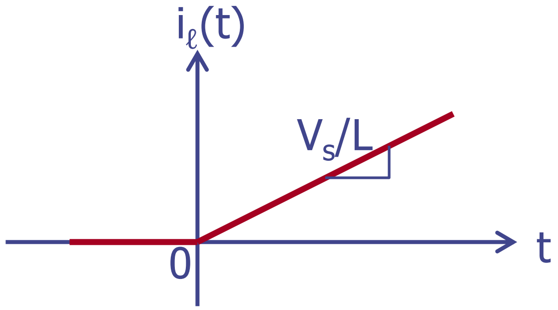
Now, consider an initially relaxed inductor L ($i_\ell(0)=0$) being charged by a constant voltage source V_s through a switch S:

$$\frac{di_\ell(t)}{dt} = \frac{V_s}{L}$$

$$i_\ell(t) = \frac{V_s}{L}t$$



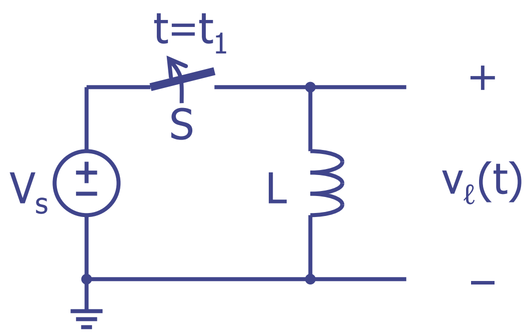
The time response of $i_o(t)$ is as shown:



Arc Discharge due to Discontinuous Inductor Current

After charging the inductor to $i_\ell(t_1^-) \neq 0$, the switch is turned off abruptly at $t=t_1$. What will happen next?

Clearly, at $t=t_1^-$, $i_\ell(t_1^-) \neq 0$. However, by switching off S, an open circuit exists between V_s and L, and



$$i_\ell(t_1^+) = 0.$$

Then

$$v_\ell(t_1) = L\frac{i_\ell(t_1^+) - i_\ell(t_1^-)}{t_1^+ - t_1^-} = -\infty$$

This large inductor voltage applied across the switch causes breakdown of the air (if the electric field strength is larger than 3V/μm), and spark occurs. This is known as arc discharge and in general should be avoided.

6.3.2 Continuity of Inductor Current

From the inductor equation:

$$v_\ell(t) = L\frac{di_\ell(t)}{dt}$$

we learn that any jump in inductor current ($\Delta i_\ell \neq 0$, but $\Delta t = 0$) will induce an infinite inductor voltage. Hence, for inductor voltage to remain finite, there can be no jump in inductor current: this is known as the continuity of inductor current, i.e. $i_\ell(0^+) = i_\ell(0^-)$.

Another way to prove the continuity of the inductor current is by the energy method. The energy stored in an inductor is

$$E = \tfrac{1}{2}LI^2$$

A sudden jump in current will lead to a sudden jump in energy, which implies infinite power. This is physically impossible.

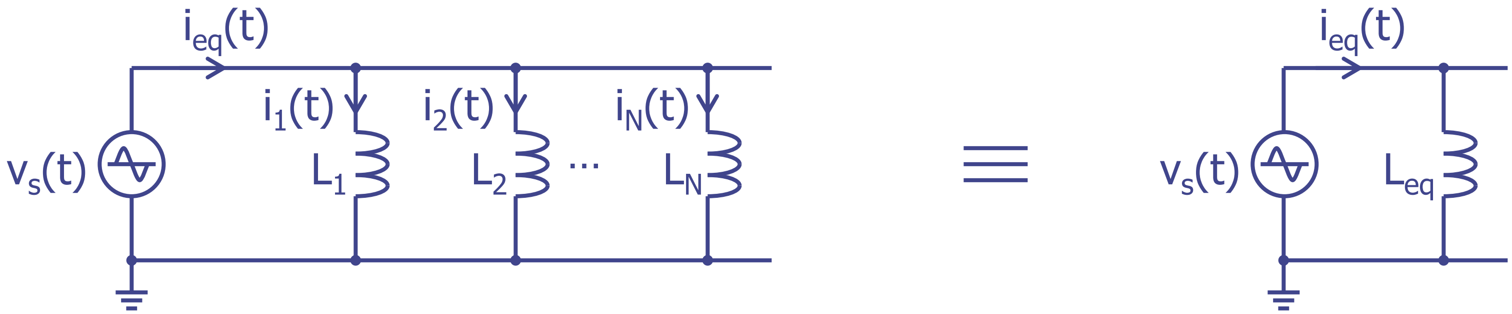
N.B.

For C, V has to be continuous but I can jump

For L, I has to be continuous but V can jump

6.3.3 Inductors in Parallel and in Series

Consider driving N inductors connected in parallel with a time varying voltage source $v_s(t)$:



$$i_{eq}(t) = i_1(t) + i_2(t) + \cdots + i_N(t)$$

$$\frac{di_{eq}(t)}{dt} = \frac{di_1(t)}{dt} + \frac{di_2(t)}{dt} + \cdots + \frac{di_N(t)}{dt}$$

$$\frac{v_s}{L_{eq}} = \frac{v_s}{L_1} + \frac{v_s}{L_2} + \cdots + \frac{v_s}{L_N}$$

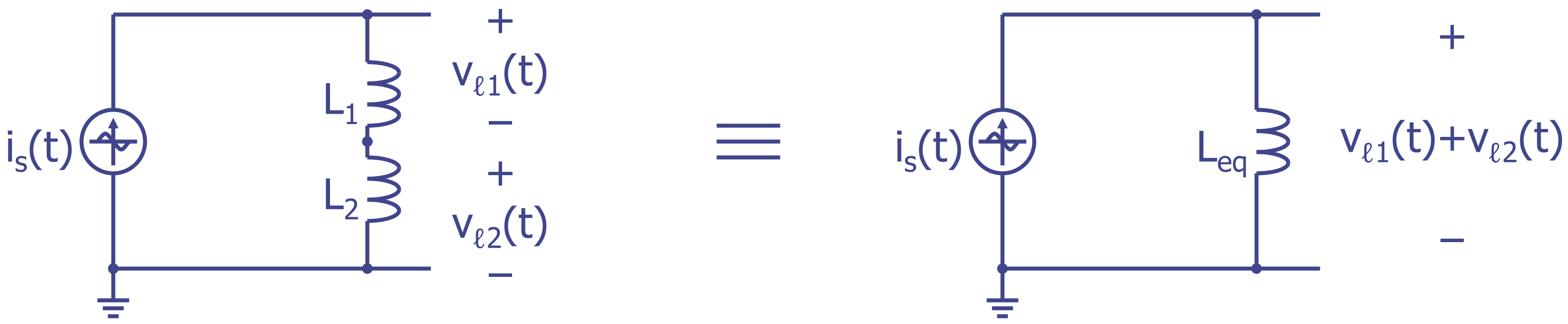
$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

Hence, for inductors connected in parallel:

$L_{eq} = L_1 \,||\, L_2 \,||\, \ldots \,||\, L_N$

Inductors in Series

Consider driving two inductors connected in series with a time varying current source $i_s(t)$:



$$v_{\ell 1}(t) + v_{\ell 2}(t) = (L_1 + L_2)\frac{di_s(t)}{dt} = L_{eq}\frac{di_s(t)}{dt}$$

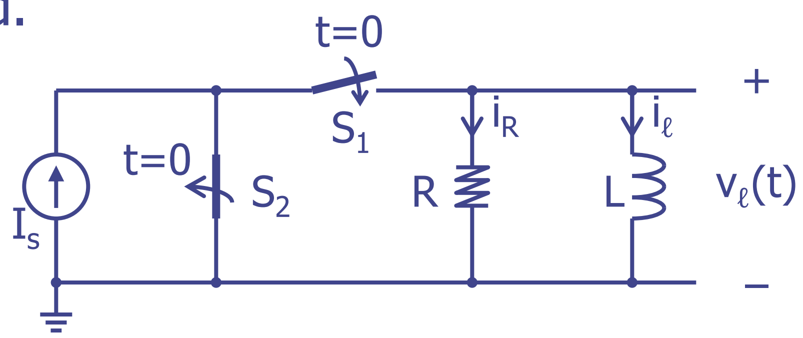
$$\Rightarrow \quad L_{eq} = L_1 + L_2$$

In general, for inductors connected in series:

$$L_{eq} = L_1 + L_2 + \ldots + L_N$$

6.3.4 Charging Inductor with I_s through R

Consider the charging of the RL circuit through a current source I_s. Note that we cannot switch open a current source, and hence, a second switch S_2 is added in parallel with I_s, and stays closed until S_1 is closed.



At t=0, S_1 is closed (and S_2 is opened). By KCL, we have

$$v_l(t) = L\frac{di_l(t)}{dt} = i_R(t)R = (I_S - i_l(t))R$$

$$\Rightarrow \qquad \frac{di_l(t)}{dt} + \frac{i_l(t)}{L/R} = \frac{I_S}{L/R}$$

$$\Rightarrow \qquad \frac{di_l(t)}{dt} + \frac{i_l(t)}{\tau} = \frac{I_S}{\tau} \qquad with \quad \tau = \frac{L}{R}$$

6.3.5 Inductor as Short Circuit in Steady State

The RL equation has the same form as the RC equation. With the inductor initially relaxed ($i_\ell(0^+)=0$), and noting that $i_\ell(\infty)=I_s$, we have

$$i_\ell(t) = I_s(1 - e^{-t/\tau})$$

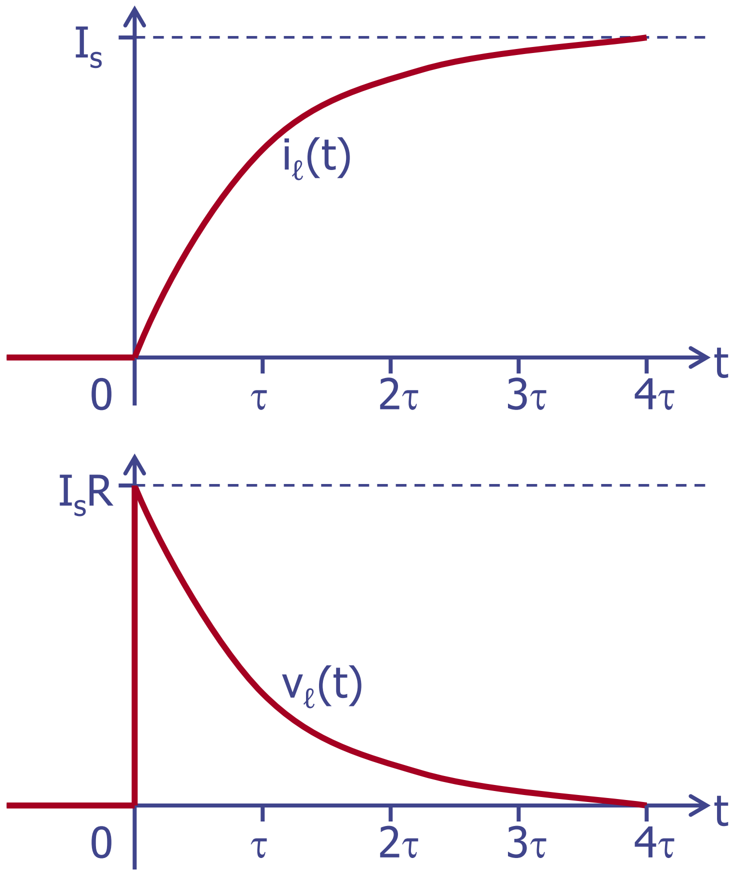
The inductor voltage is given by

$$v_\ell(t) = L\frac{di_\ell(t)}{dt} = I_s R \times e^{-t/\tau}$$

In the initial state ($t=0^+$), with continuity of the inductor current $i_\ell(0^+)=i_\ell(0^-)=0$.

In the final state ($t=\infty$), the inductor behaves as a short circuit (for $v_\ell(\infty)=0$).

Note that the inductor voltage can be discontinuous.



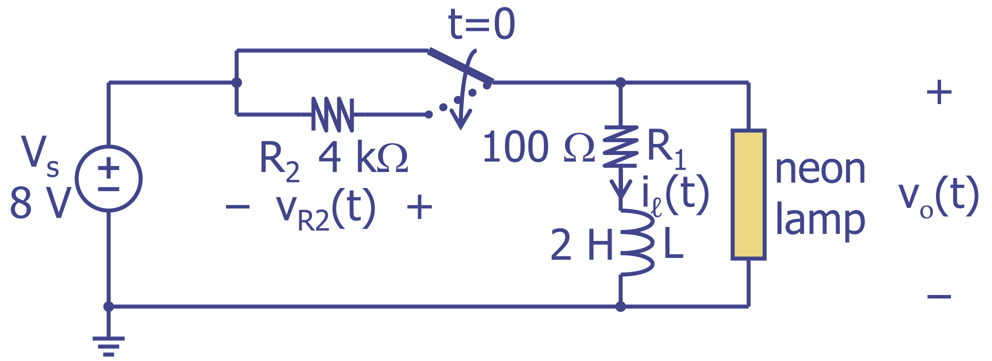
Applications of RL circuits

In many applications, such as fluorescent lamps and neon lamps, a very high voltage is needed for start up. We may make use of the continuity of inductor current to generate a high voltage for this purpose.

Ballast

Example 6-12

Example 6-12: Discuss the switching action of the following circuit that generates a high voltage of over 300 V from a low voltage source to ignite a neon lamp.



Soln.:

Prior to switching, the circuit is in the steady state, with L being a short circuit ($v_\ell(0^-)=0$), and

$$i_\ell(0^-) = \frac{V_s}{R_1} = 80\text{mA}$$

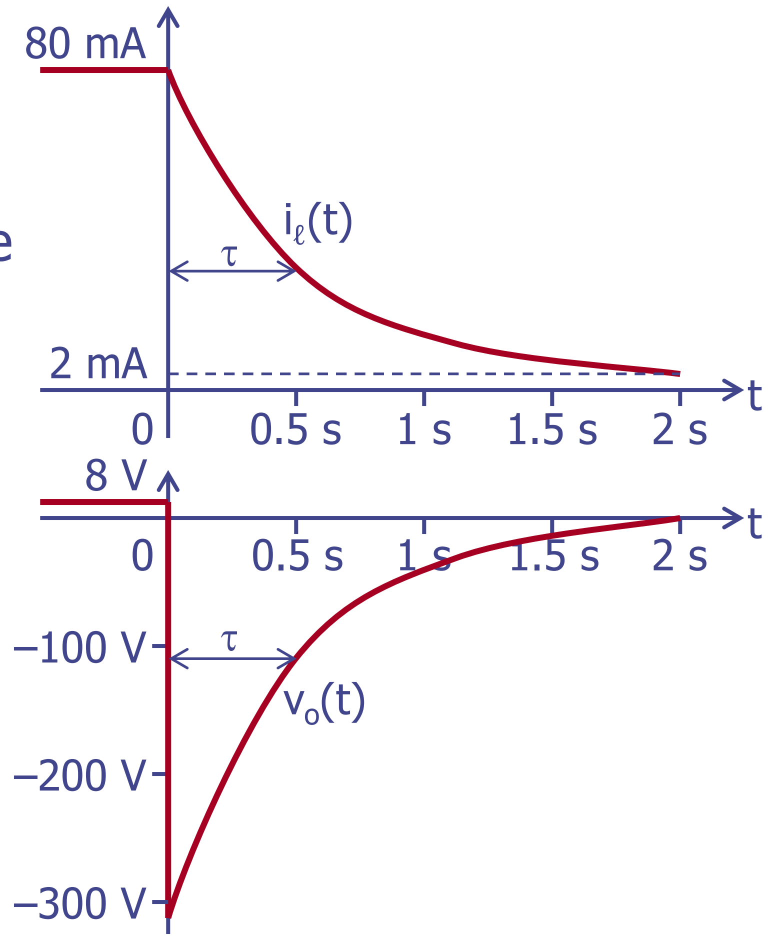
Note that the neon lamp remains an open circuit before it is ignited.

Example 6-12 (cont.)

Right after switching, the inductor current cannot change instantaneously, and $i_\ell(0^+)$=80 mA flows through R_2, and

$$v_{R_2}(0^+) = -80m \times 4k = -320V$$

$$\Rightarrow \quad v_o(0^+) = v_{R_2}(0^+) + V_s = -312V$$



This 312 V is high enough to ignite the neon lamp. The time constant is $\tau = L/(R_1+R_2)$ = 0.488 ms. Eventually, as $t \to \infty$, the inductor becomes a short circuit again, and

$$i_\ell(\infty) = \frac{V_s}{R_1 + R_2} \approx 2mA$$

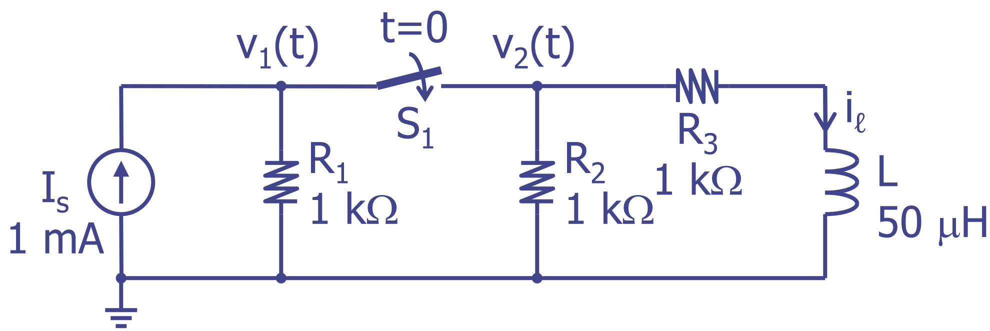
Hence,

$$i_l(t) = 2m + (80m - 2m) \times e^{-t/0.4888m}$$

$$= 2m + 78m \times e^{-t/0.4888m}$$

Example 6-13

Example 6-13: Find and sketch $v_1(t)$ and $i_\ell(t)$ for the circuit below.



Soln.:

Prior to switching,

$v_1(0^-) = I_s R_1 = 1$ V

$i_\ell(0^-) = 0$ A (and $v_2(0^-) = 0$ V)

At the instant of switching, with continuity of the inductor current,

and $i_\ell(0^+) = i_\ell(0^-) = 0$ A

$v_1(0^+) = I_s(R_1||R_2) = 1\text{m}\times 500 = 0.5$ V

For more complicated RC and RL circuits, the time constant is given by $\tau = R_{eq}C$ or L/R_{eq}, where R_{eq} is the *equivalent resistance* as seen by the capacitor or inductor.

Example 6-13 (cont.)

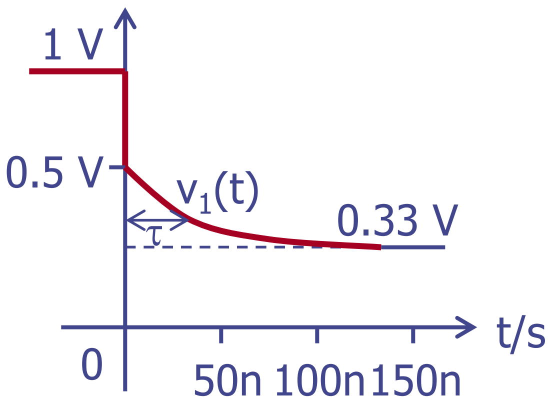
At $t \to \infty$, L behaves as a short circuit, and

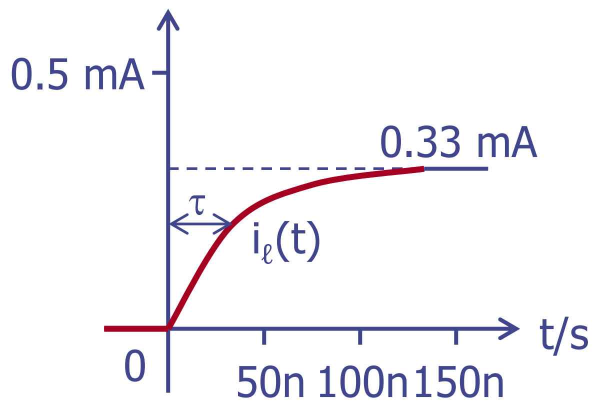
$$i_\ell(\infty) = I_s(R_1||R_2)/((R_1||R_2)+R_3)$$
$$= 1m \times 0.5k/1.5k$$
$$= 0.333 \text{ mA}$$

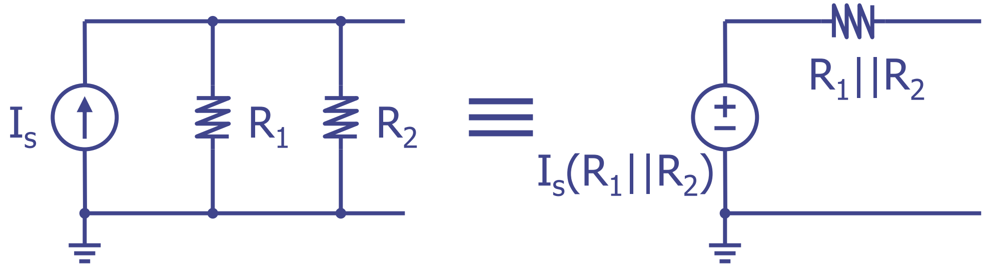
$$v_1(\infty) = I_s(R_1||R_2||R_3)$$
$$= 1m \times 333$$
$$= 0.333 \text{ V}$$

Use equivalent resistance to compute the time constant:

$$\tau = L/((R_1||R_2)+R_3)$$
$$= 33.3 \text{ ns}$$

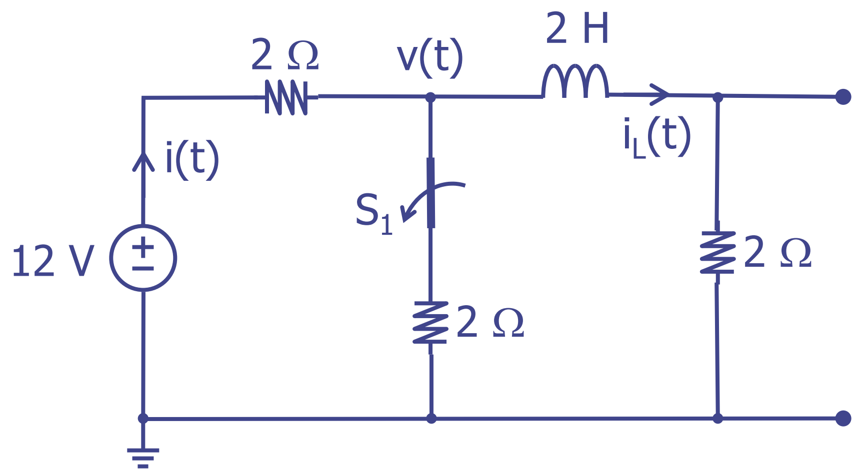




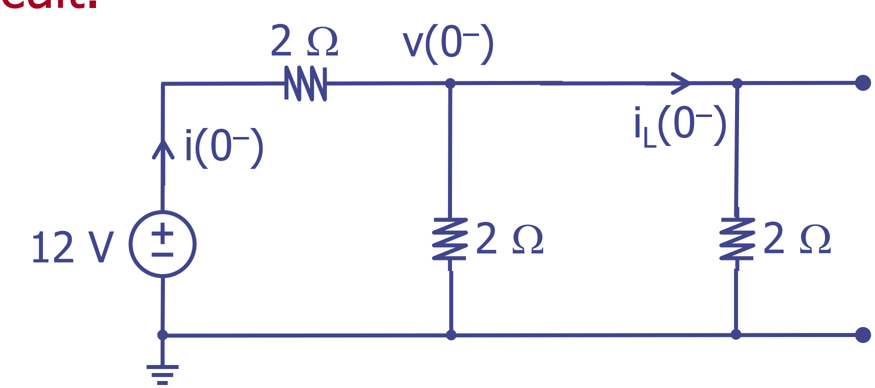


Example 6-14

Example 6-14: For the circuit in below, find the expressions for and sketch v(t) and i(t) before and after the switch is opened.



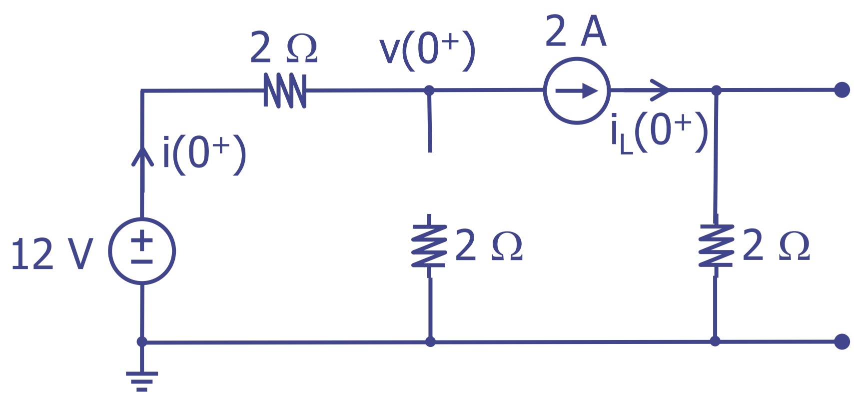
(i) Before the switch is opened, the inductor behaves as a short circuit.



$v(0^-) = 4$ V
$i(0^-) = 4$ A
$i_L(0^-) = 2$ A

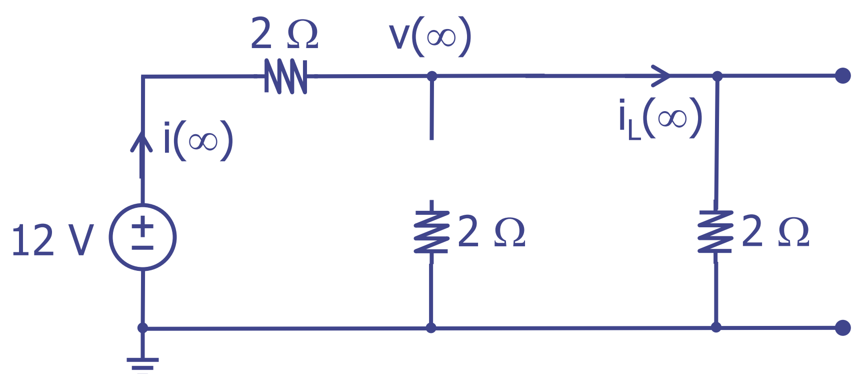
Example 6-14 (cont.)

(ii) $t=0^+$ scenario: the inductor current can't change instantly. Therefore $i_L(0^+) = 2$ A and the inductor momentarily behaves as a 2-A current source as shown in below.



$v(0^+) = 8$ V
$i(0^+) = 2$ A
$i_L(0^+) = 2$ A

(iii) At $t \rightarrow \infty$: the inductor behaves as a short circuit again.



$v(\infty) = 6$ V
$i(\infty) = 3$ A
$i_L(\infty) = 3$ A

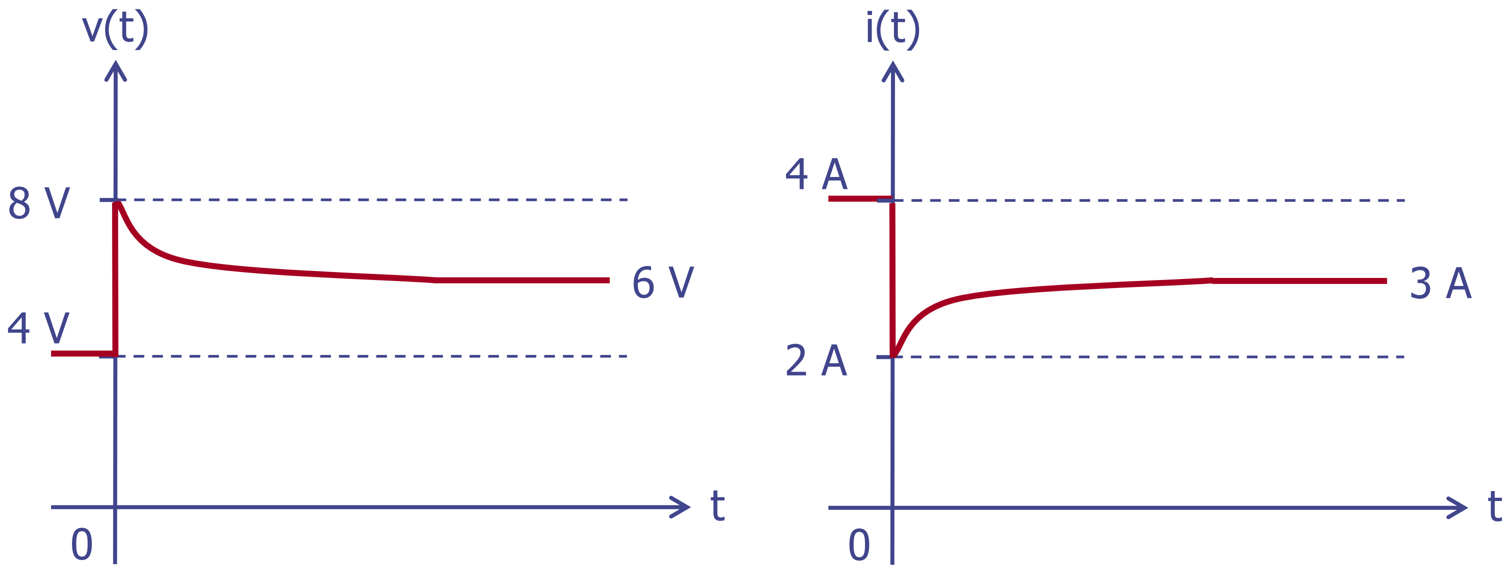
Example 6-14 (cont.)

After the switch is opened, the equivalent resistance seen by the inductor is 2 Ω + 2 Ω.

Time constant: τ = 2 H/4 Ω = 0.5 s

Therefore

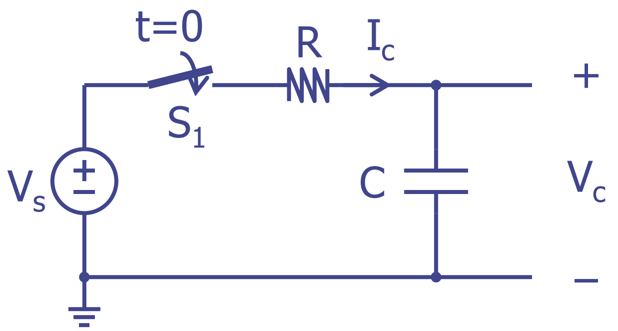
$$v(t) = 6 + 2e^{-t/0.5}$$
$$i(t) = 3 - e^{-t/0.5}$$



Transfer Function Approach (RC Circuit)

Capacitor connected to a series resistor excited by a voltage source, i.e., the Thevenin's equivalent circuit.

Transfer functions:



$$\frac{I_c(s)}{V_s(s)} = \frac{V_s(s)/(R+\frac{1}{sC})}{V_s(s)}$$

$$= \frac{1}{R+\frac{1}{sC}} = \frac{sC}{1+sRC} = \frac{sC}{1+s\tau}$$

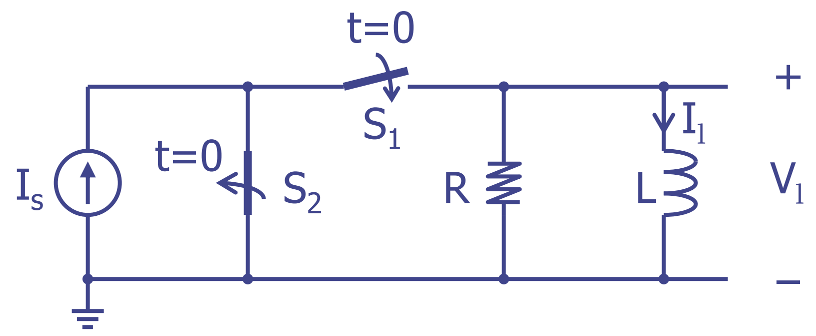
$$\frac{V_c(s)}{V_s(s)} = \frac{I_c(s)/(sC)}{V_s\ (s)} = \frac{1}{1+s\tau} \qquad (\tau = RC)$$

The numerator and denominator polynomials in s in the transfer function are of first degree at most. Hence first order system.

The transient analysis can also be solved by Laplace Transform techniques (not taught in this class).

Transfer Function Approach (RL Circuit)

Inductor connected to a parallel resistor excited by a current source, i.e., the Norton's equivalent circuit.



Transfer functions:

$$\frac{V_l(s)}{I_s(s)} = \frac{I_s(s)(R \parallel sL)}{I_s(s)}$$

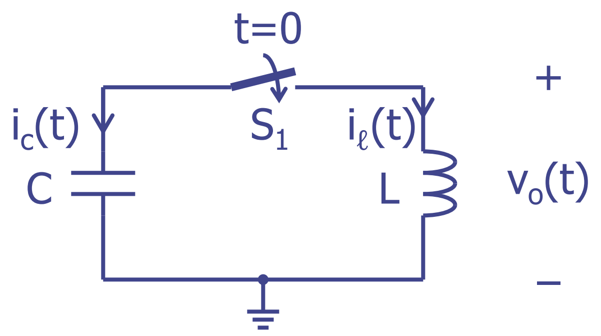
$$= \frac{sRL}{R + sL} = \frac{sL}{1 + s\frac{L}{R}} = \frac{sL}{1 + s\tau}$$

$$\frac{I_l(s)}{I_s(s)} = \frac{V_l(s)/(sL)}{I_s\ (s)} = \frac{1}{1 + s\tau} \qquad (\tau = \frac{L}{R})$$

The equations are in the same form as the RC circuit.

LC Resonator
(A Second Order System)

Assume the capacitor C is initially charged to $v_c(0^-) = V_a$. At t=0, it is connected in parallel to an inductor L that is initially relaxed ($i_\ell(0^-) = 0$). Discuss the subsequent action.



When S_1 is closed, $v_c(t) = v_\ell(t) = v_o(t)$, and $i_c(t) = -i_\ell(t)$. Now,

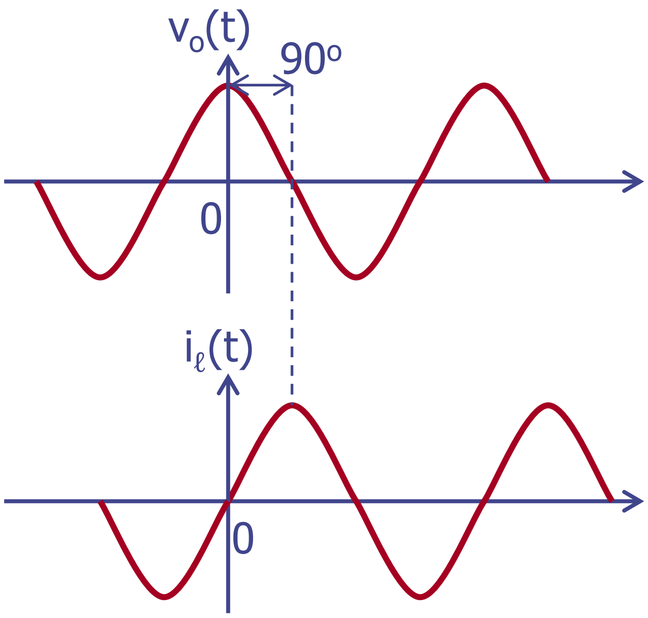
$$v_o(t) = L\frac{di_\ell(t)}{dt} = L\frac{d}{dt}\left(-C\frac{dv_o(t)}{dt}\right)$$

$$\Rightarrow \qquad \frac{d^2v_o(t)}{dt^2} + \frac{1}{LC}v_o(t) = 0$$

LC Resonator (cont.)

Define $\omega_o^2 = 1/(LC)$, and the equation of a simple harmonic oscillator is obtained:

$$\frac{d^2 v_o(t)}{dt^2} + \omega_o^2 v_o(t) = 0 \qquad \omega_o = \frac{1}{\sqrt{LC}}$$



The general solution (by inspection) is

$$v_o(t) = a\cos(\omega_o t) + b\sin(\omega_o t)$$

By matching the initial conditions, we have

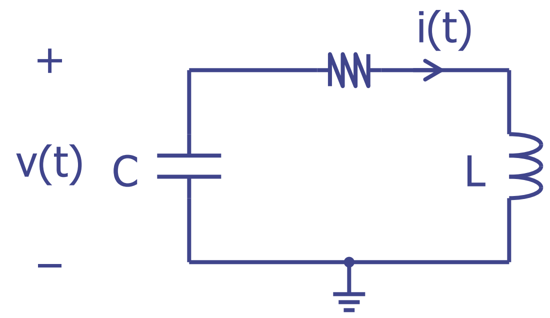
$$v_o(t) = V_a \cos(\omega_o t)$$

$$i_\ell(t) = -C\frac{dv_o(t)}{dt} = \omega C V_a \sin(\omega_o t)$$

The electrical energy stored in the capacitor is gradually converted to the magnetic energy of the inductor, then back and forth forever. The circuit keeps oscillating at the resonance frequency ω_o and is thus known as an LC resonator.

RLC Circuits
(A Second Order System)

The flash lamp discharge circuit can be modeled more accurately as an RLC circuit. The series RLC circuit is shown below.



By KVL,

$$v(t) = i(t)R + L\frac{di(t)}{dt}$$

$$= -C\frac{dv(t)}{dt}R - LC\frac{d^2v(t)}{dt^2}$$

$$\Rightarrow \quad \frac{d^2v(t)}{dt^2} + \frac{R}{L}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

RLC Circuit (cont.)

The second order ODE can be more conveniently written in terms of the resonance frequency ω_o and the damping ratio ζ:

$$\frac{d^2v(t)}{dt^2} + 2\zeta\omega_o\frac{dv(t)}{dt} + \omega_o^{\ 2}v(t) = 0 \qquad \omega_o = \frac{1}{\sqrt{LC}} \qquad \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

The general shapes of damped harmonic oscillation are shown below: